A New Method in Analytic Geometry.

By William E. Story.

About a year ago, in connection with a course of lectures on plane cubic curves, I had occasion to prove that the four tangents to a non-singular plane cubic from any point of the same have a constant anharmonic ratio, and an instantaneous proof then occurred to me which is certainly, having a purely algebraic basis, not open to the objection which Sturm* has raised (without any real foundation, as it seems to me) to that given by Salmon. Shortly afterwards I applied the same method to the proof of other geometrical theorems, and gave it a general form in my own mind, at least, but have only recently found leisure to put it into shape for publication. It may not be uninteresting as an application to geometry of the fundamental theorem of algebra briefly stated: every equation has a root.

Let $f(\alpha, x) = 0$ be an equation concerning whose left member $f(\alpha, x)$ we make the following assumptions:

a). It *certainly* involves a variable α, being a rational algebraic polynomial in α.

b). It *possibly* involves a second variable x, and if so, is a rational algebraic polynomial in x.

c). It has no factor involving x but not α.

Under these assumptions $f(\alpha, x)$ may break up into factors, some of which involve α alone, and some both α and x, but this is of no consequence.

There are then two cases:

I. If the equation involve both α and x, for every assumed value of x it defines a certain finite number (its degree in α) of values of α, which may be

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* Crelle's Journal, Vol. 90, p. —.

considered so many functions of x, some of which will necessarily vary with the assumed value of x, and for every assumed value of α it determines a certain finite number (its degree in x) of values of x.

II. If the equation involve α but not x, it defines a certain finite number (its degree in α) of values of α, which are the same, whatever value x may be assumed to have, *i. e.* which are constant.

Here we make no distinction between real and imaginary values. If, then, it is known that the equation will not be satisfied by any real or imaginary value of x when a certain value is assigned to α, it must come under the second case, and the values of α determined by it are constant.

The following extension of this theorem is evident. If a quantity α is connected with k other quantities $x, y, z, \ldots.$ by an equation $f(\alpha, x, y, z, \ldots.) = 0$, say $f = 0$, whose left member is a rational algebraic polynomial in these quantities (certainly involving α and possibly $x, y, z, \ldots.$), and if $x, y, z, \ldots.$ are connected among themselves by $k - 1$ auxiliary equations $\phi = 0$, $\psi = 0, \ldots.$; then, if the equations $f = 0$, $\phi = 0$, $\psi = 0, \ldots.$ cannot be satisfied by any set of real or imaginary values of $x, y, z, \ldots.$ when a certain value is assigned to α, the values of α determined by $f = 0$ will be constant, whatever values may be assigned to $x, y, z, \ldots.$ satisfying the auxiliary equations. For, by means of the auxiliary equations, $k - 1$ of the quantities $x, y, z, \ldots.$ may be eliminated from the equation $f = 0$, which is thus reduced to an equation between α and one of the other quantities, say x, to which the previous theorem is applicable.

This extended theorem can be applied to the proof of numerous geometrical theorems. Let $x, y, z, \ldots.$ be the co-ordinates of a variable element (point, straight line or plane) of a one-way algebraic locus, *i. e.* point of a plane or twisted curve, tangent of a plane or twisted curve, generator of a ruled surface, edge of a cone, tangent plane of a cone or developable surface; let $\phi = 0$, $\psi = 0, \ldots.$ be the equations of the locus, together with whatever identities exist between the co-ordinates (*e. g.* such an identity exists between the six co-ordinates of a straight line); and let there be a geometrical magnitude α determined by any position of the variable element and, it may be, certain fixed elements or geometrical forms (curves and surfaces), such that α is capable of determination by an algebraic equation which, when rationalized and cleared of fractions, is $f(\alpha, x, y, z, \ldots.) = 0$, say of the degree ν in α. This rationalized equation defines ν geometrical magnitudes determined by each element of the

locus, or say α stands in any one of ν geometrical relations to a given element $x, y, z, \ldots$. If, then, it is found that a certain value stands in neither of these relations to any real or imaginary element of the locus, the above theorems show that the several geometrical magnitudes determined by an element of the locus are constant; *i. e.* that, taken in some order or other, they have the same values for all elements of the locus. This will be more evident from the examples which I give below. To prove a given theorem involving the constancy of a magnitude α, all that has to be done is to find a value of α which can be shown to be impossible for any real or imaginary element of the locus, and the readiest manner of doing this seems to be to assume some simple value, such as 0 or ∞, and prove its impossibility. In a large number of cases the very nature of the locus will make this certain, and in these cases this method is instantaneous, furnishing, it seems to me, the simplest conceivable proof. In other cases the assumed value of α, say 0 or ∞, will be possible only when the expression of α assumes an indeterminate form, and then it will be necessary to show that the true value is not the assumed value. The first three applications given below belong to the first class, the others to the second class of cases. In both these classes of cases α can be expressed (not necessarily rationally) in terms of the co-ordinates of the element of the locus. It would be interesting to find a case in which α could not be so expressed, but to which the method is still applicable.

The auxiliary equations may evidently constitute a restricted system, as in the case of the twisted cubic curve, without affecting the applicability of the method.

From the readiness with which one application suggests another, it seems probable that the method may be useful in the discovery of new theorems.

It may be useful here to give an example to which the method is not applicable, for the purpose of showing the cause of failure. That the sum of the focal distances of a point of an ellipse is constant cannot be so proved, for the *rationalized* equation which determines the sum of these distances determines also their difference, and if the sum is constant, the difference will vary, and *vice versa*, and it will not be possible to show that neither sum nor difference can have a certain value, say 0; in fact, the ordinary method of proof of the theorem stated holds only for *real* points of the ellipse, while the curve defined as the locus of points the sum of whose distances from two given points is a given constant will be an ellipse only if that constant is greater than the distance

between the given points. In a certain sense we may say that, for a part of the curve, including the real part, the sum, and for the rest the difference, of the focal distances is constant.

One of the simplest cases is that in which α is the anharmonic ratio of four points, lines or planes determined by the variable element of the locus. There are six values of this anharmonic ratio, depending upon the order in which the four points, lines or planes are taken, so that the order of the rationalized equation $f = 0$ is 6; and to prove the constancy of these values for all elements of the locus by our method, it would be necessary to show that there is a value which *neither* of these anharmonic ratios can have for any real or imaginary element of the locus.

As a matter of form I call attention here, once for all, to the evident fact, already mentioned incidentally, that in all the following applications the quantity α, whose constancy is to be proved, is connected with the co-ordinates of the variable element of the given locus by a rational algebraic equation, as is necessary for the proof.

1. The anharmonic ratio α of the junctions of a variable point of a conic to four fixed points of the same, taken in a given order, is constant. For it can have the value 0 only if two of the junctions coincide, *i. e.* only if the variable point lies on the junction of two of the fixed points; but, the curve being of the second order, the two fixed points are the only points of the conic on their junction; therefore $\alpha = 0$ only if the variable point coincides with one of the fixed points; but then one of the four junctions in question is the tangent at this fixed point and the other three junctions are lines differing from each other and from this tangent; so that, even in this case, α is not 0, and therefore is constant for every position of the variable point.

2. The four tangents to a non-singular plane cubic curve from any point of the same (other than the two coincident tangents at the point) have a constant anharmonic ratio α (*i. e.* there are six constant values of α, according to the order in which the tangents are taken). For $\alpha = 0$ only when two of these tangents coincide, which they never do; for the curve, being of the third order, has no double tangent, and an inflexional tangent meets the curve only at its point of contact and counts for only one of the four tangents from this point.

3. The four planes joining a variable tangent of a twisted cubic to four fixed points of the same, taken in a given order, have a constant anharmonic ratio α,

For $\alpha = 0$ only if two of these planes coincide, *i. e.* only if the tangent lies in the same plane with two of the fixed points; this plane will then meet the curve in the two fixed points and in two points at the contact of the tangent, *i. e.* in four points in all, which is impossible, the curve being of the third order; or the point of contact of the tangent will be one of the fixed points, in which case one of the planes is the osculating plane at this point and does not coincide with either of the other three planes.

The reciprocals of these three theorems are known to be true by the principle of duality. These reciprocals are:

The anharmonic ratio α of the intersections of a variable tangent of a conic with four fixed tangents of the same, taken in a given order, is constant.

The four intersections of a non-singular plane curve of the third class (which is of the sixth order) with any tangent of the same (other than two coincident intersections at the point of contact of the tangent) have a constant anharmonic ratio α (*i. e.* there are six constant values of α, according to the order in which the points are taken).

The four points of intersection of a variable generator of a developable surface of the third class with four fixed tangent planes of the same, taken in a given order, have a constant anharmonic ratio α.

4. The angle between the junctions of a variable point of a circle with two fixed points of the same, taken in a given order, is constant. Let $P(x, y)$ be the variable point, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ the fixed points, ϑ the angle which PP_2 makes with PP_1, r the radius of the circle, and the centre of the circle in the origin of co-ordinates; then $x^2 + y^2 = r^2$,

$$\tan\vartheta = \frac{(y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1}{x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + x_1x_2 + y_1y_2},$$

which can be 0 only when P lies on the junction of P_1P_2, or when P is infinite (*i. e.* one of the circular points). If P lies on P_1P_2 it must coincide with P_1 or P_2, since these are the only points of the circle on this line; but if it coincides with P_1 the junctions in question are the tangent at P_1 and the line P_1P_2, and the tangent of the angle between these lines is not 0. If P is a circular point, $y = \pm ix$, $x = \infty$,

$$\tan\vartheta = \frac{y_2 - y_1 \mp i(x_2 - x_1)}{x_1 + x_2 \pm i(y_1 + y_2)},$$

provided P_1 and P_2 are finite, and this value of $\tan\vartheta$ is evidently not 0. If P or P_2 is a circular point, evidently $\tan\vartheta = i$.

5. The segment cut out of a variable tangent to a conic by two fixed tangents of the same subtends a constant angle at either focus. This is, as is well known, readily proved by the reciprocal of number 1 above, but it may be proved independently as follows: Let P be the focus, P_1 and P_2 the intersections of the variable tangent with the two fixed tangents, and ϑ the subtended angle; then, from the general expression given in the last number, it is evident that $\tan\vartheta$ will be 0, since P is finite, only if P, P_1 and P_2 lie in one straight line, *i. e.* only if the variable tangent passes through the intersection of the fixed tangents, and therefore coincides with one of them, in which case the angle ϑ is the angle between the lines joining P to the intersection of the fixed tangents and to the point of contact of one of them, whose tangent is evidently not 0. If one of the fixed tangents passes through the focus, $\tan\vartheta$ is evidently constantly $= i$.

6. The ratio of the distances of a variable point of a conic from either focus and the corresponding directrix is constant. We define the focus as the intersection of two circular tangents, *i. e.* tangents each of which passes through a circular point, and the corresponding directrix as its polar, *i. e.* the chord of contact of tangents from it. Evidently this ratio will be 0 only when the distance from the focus is 0 or that from the directrix is infinite, *i. e.* only when the variable point is the point of contact of one of the tangents from the focus, or when the variable point is infinite. If the variable point is infinite, the ratio in question is evidently the cosecant of the angle between the directrix and the direction of the infinite point, *i. e.* is not in general 0. If the variable point is the point of contact of one of the tangents from the focus, then its focal distance is 0, as is also its distance from the directrix on which it lies, and the ratio in question is undetermined. For a point very near the point of contact, both distances are small of the same order and their ratio is determinate. Let the origin be the focus and axis of x perpendicular to the directrix, whose equation is then of the form $x + d = 0$; the point $(-d, id)$ is then the point of contact of one of the tangents from the focus, and the ratio in question is $\frac{\sqrt{x^2+y^2}}{x+d}$ for this point. For a neighboring point $x = -d + \delta$, $y = id + \left(\frac{\partial y}{\partial x}\right)\delta + \frac{1}{2}\left(\frac{\partial^2 y}{\partial x^2}\right)\delta^2$, where $\left(\frac{\partial y}{\partial x}\right)$ and $\left(\frac{\partial^2 y}{\partial x^2}\right)$ are to be taken for the point of contact. Evidently $\left(\frac{\partial y}{\partial x}\right) = -i$, and $\left(\frac{\partial^2 y}{\partial x^2}\right)^2$ has a value different from 0, if the point of contact is finite. For this

neighboring point then $y = i(d - \delta) + \frac{1}{2}\left(\frac{\partial^2 y}{\partial x^2}\right)\delta^2$,

$$x^2 + y^2 = \left(\frac{\partial^2 y}{\partial x^2}\right)\delta^2\left[i(d-\delta) + \frac{1}{4}\left(\frac{\partial^2 y}{\partial x^2}\right)\delta^2\right], \; x + d = \delta,$$

and in the limit, as δ vanishes,

$$\frac{\sqrt{x^2+y^2}}{x+d} = \sqrt{id\left(\frac{\partial^2 y}{\partial x^2}\right)},$$

i. e. is not 0. If the conic is a circle, $d = \infty$, and for every finite point $x + d$ is infinite and $\sqrt{x^2+y^2}$ is finite, so that the ratio in question is *constantly* 0.

Of course this is not the shortest proof of the last theorem. Indeed, the definitions of focus and directrix show that the equation of the conic, with the same choice of co-ordinates as before, will be of the form

$$x^2 + y^2 - e^2(x+d)^2 = 0,$$

where e is a constant, and we have directly, for any point (x, y) of the curve,

$$\frac{\sqrt{x^2+y^2}}{x+d} = \pm e.$$

For a circle $d = \infty$, $e = 0$, $de = r$, *where* r *is the radius.*

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